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Eigenvalues for a one-step process in one dimension

A Malaspinas

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

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Abstract. The eigenvalues of a master operator W_β defined by

$$\dot{P}_n(t) = (W_\beta P(t))_n - P_n(t)$$
$$= \frac{1}{e^{\beta(n-1)}+1} P_{n-1}(t) + \frac{1}{e^{-\beta(n+1)}+1} P_{n+1}(t) - P_n(t), \qquad n \in \mathbb{Z}, \beta > 0,$$

and its symmetric counterpart are found.

1. Introduction

In this work we will discuss a one-dimensional non-linear one-step process defined by the master equation

$$\dot{P}_n(t) = (W_\beta P(t))_n - P_n(t)$$
$$= \frac{1}{e^{\beta(n-1)}+1} P_{n-1}(t) + \frac{1}{e^{-\beta(n+1)}+1} P_{n+1}(t) - P_n(t). \tag{1}$$

This equation describes a random walk in a symmetric potential which may be attractive ($\beta > 0$), flat ($\beta = 0$) or repulsive ($\beta < 0$).

At this stage the interest of (1) is rather academic in the sense that we can exactly calculate the eigenvalues of a non-linear problem, which is rare enough to be worth mentioning. These eigenvalues are calculated for the symmetric counterpart of the operator W_β and it is of some pedagogical interest to compare them with the eigenvalues of W_β itself. It is noteworthy that the Fokker-Planck approximation of (1) can be exactly diagonalised and hence comparisons between the two problems can be made (M Hongler, private communication).

Note finally that the 'fermionic' character of the transition rates in (1) indicates that this equation may find some application in problems related to coding or biological processes.

From now on we will restrict ourselves to the attractive case $\beta > 0$ and will discuss W_β as an operator on the real l_2. In § 2 we will calculate the stationary state of (1) and use it to symmetrise W_β. In § 3 we will diagonalise the symmetric counterpart of W_β and finally exhibit the eigenvalues of W_β itself.

2. The stationary solution

Introduce $a > 1$ by $\log a^2 = \beta$. Then (1) takes the form

$$\dot{P}_n(t) = (T_a P(t))_n - P_n(t)$$
$$= r_{n+1} P_{n+1}(t) + g_{n-1} P_{n-1}(t) - (r_n + g_n) P_n(t) \tag{2}$$

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with $r_n(a)=(a^{-2n}+1)^{-1}$, $g_n(a)=(a^{2n}+1)^{-1}$ and $r_n(a)+g_n(a)\equiv 1$. The symmetry of the problem is reflected in the fact that $r_{-n}=g_n$, $n\in Z$. It is easy to find the stationary states for equation (2) (van Kampen 1981). One has to solve

$$r_n P^s_n - g_{n-1}P^s_{n-1} = -J \tag{3}$$

with J an arbitrary constant. It is easily checked that the only solution of (3) satisfying $\Sigma^{\infty}_{n=-\infty} P^s_n < \infty$ is the one corresponding to $J=0$, and one finds

$$P^s_n = \frac{1}{N} a^{-n^2}\cosh(n\log a), \qquad N=\sum_{n=-\infty}^{\infty} a^{-n^2}\cosh(n\log a), \tag{4}$$

which is symmetric as expected. This is just the stationary state of Alkemade's diode (van Kampen 1961). For $J=0$, (3) is just the detailed balance condition and P^s_n can be used to symmetrise T_a by defining

$$P_{n(t)} = (P^s_n)^{1/2} Q_n(t). \tag{5}$$

Using the notation $s(n):=\sinh(n\log a)$, $c(n)=\cosh(n\log a)$ and $t(n)=\tanh(n\log a)$, equation (2) becomes

$$\begin{aligned}\dot{Q}_n(t) &= (S_a Q(t))_n - Q_n(t)\\ &= \frac{a^{1/2}}{2[c(n-1)c(n)]^{1/2}} Q_{n-1}(t)\\ &\quad + \frac{a^{1/2}}{2[c(n)c(n+1)]^{1/2}} Q_{n+1}(t) - Q_n(t).\end{aligned} \tag{6}$$

Note that the eigenvalues of S_a are eigenvalues of T_a but the contrary need not be true since the transformation (5) has an unbounded inverse.

3. Eigenvalues of S_a

It is easy to see that S_a has a pure point spectrum. Indeed its Hilbert-Schmidt norm is given by

$$H(S_a) = \sum_{n=0}^{\infty} \frac{a}{c(n)c(n+1)} \tag{7}$$

which is clearly finite and hence S_a is compact. Note one more easily checked property of S_a: if E^{λ} is an eigenvector with eigenvalue λ then $E^{-\lambda}$ defined by $E^{-\lambda}_n = (-1)^n E^{\lambda}_n$ is an eigenvector with eigenvalue $-\lambda$.

Let $\sigma_p(S_a)$ be the set of eigenvalues of S_a. We will sketch the proof that $\sigma_p(S_a) = \{\pm 1/a^k\}_{k=0}$.

Step 1. $\{\pm 1/a^{2L}\}_{L=0} \subset \sigma_p(S_a)$, by constructing eigenvectors for these values.

Step 2. $\{\pm 1/a^{2L+1}\}_{L=0} \subset \sigma_p(S_a)$, by constructing eigenvectors for these values.

Step 3. $0 \notin \sigma_p(S_a)$.

Step 4. $\{\pm 1/a^k\}_{k=0}$ are all the eigenvalues and are non-degenerate.

Proof of step 1. In view of the properties of S_a stated at the beginning of this section it is enough to show that $\{1/a^{2L}\}_{L=0} \subset \sigma_p(S_a)$. We define e^L by

$$e_n^L = \sum_{k=0}^{L} \mu_{2k}(L) s(n)^{2k}. \tag{8}$$

It is enough to show that there exist coefficients $\{\mu_{2k}(L)\}_{k=0}^{L}$ such that E^L defined by

$$E_n^L = (P_n^s)^{1/2} e_n^L \tag{9}$$

is an eigenvector of S_a with eigenvalue $1/a^{2L}$ and hence prove step 1.

By replacing E^L in $S_a E^L = (1/a^{2L}) E^L$ we find

$$(T_a^* e^L)_n = \tfrac{1}{2}(1 + t(n)) e_{n-1}^L = (1/a^{2L}) e_n^L,$$

where T_a^* is the adjoint of T_a defined in equation (2). Inserting (8) we find

$$\sum_{k=0}^{L} \{\mu_{2k}(L)[\tfrac{1}{2}(1+t(n))s(n-1)^{2k} + \tfrac{1}{2}(1-t(n))s(n+1)^{2k}]\}$$

$$= \frac{1}{a^{2L}} \sum_{k=0}^{L} \mu_{2k}(L) s(n)^{2k}. \tag{10}$$

Using $s(n \pm 1) = s(n)c(1) \pm s(1)c(n)$, $c(n)^2 = 1 + s(n)^2$, some algebra and summation index manipulation, one shows that the determinant of this system is just

$$\prod_{p=0}^{L} \left(\frac{1}{a^{2p}} - \frac{1}{a^{2L}} \right) = 0$$

which proves the existence of the $\mu_{2p}(L)$. Evidently $E_n^L = (P_n^s)^{1/2} e_n^L$ defines a vector $E^L \in l_2$ and hence step 1 is proven.

Proof of step 2. Again it is enough to prove $\{1/a^{2L+1}\}_{L=0} \subset \sigma_p(s_a)$. It is again enough to show that the vector O^L defined by

$$O_n^L = (P_n^s)^{1/2} o_n^L$$

with

$$o_n^L = \sum_{k=0}^{L} \mu_{2k+1}(L) s(n)^{2k+1}$$

is an eigenvector with eigenvalue $1/a^{2L+1}$. The proof follows the same lines as in step 1.

Proof of step 3. Suppose $E^0 \in l_2$, $E^0 \neq 0$ exists such that $S_a E^0 = 0$. Define e^0 by

$$E_n^0 = (P_n^s)^{1/2} e_n^0.$$

Then e^0 must satisfy

$$e_n^0 = -a^{2(n-1)} e_{n-2}^0.$$

It is enough to concentrate on $n \geq 0$. From the recurrence relation it follows that $e_{2n}^0 = (-1)^n a^{2n^2} e_0^0$ and that $e_{2n+1} = (-1)_n a^{2n^2+2n} e_1^0$ and hence $E^0 \notin l_2$.

Proof of step 4. Since S_a is symmetric its Hilbert-Schmidt norm must be equal to the sum of the squares of its eigenvalues counted with their multiplicity. It follows that it is enough to prove (see equation (7))

$$\sum_{n=0}^{\infty} \frac{a}{c(n)c(n+1)} = 2 \sum_{k=0}^{\infty} \frac{1}{a^{2k}} = \frac{2}{1-a^{-2}},$$

which is true by elementary manipulations.

This completes the proof that $\sigma_p(S_a) = \{\pm 1/a^k\}_{k=0}$.

Let us now consider the eigenvalues of T_a, equation (2). One can show that every $\lambda \in [-1, 1]$ is an eigenvalue of T_a. Since $\lambda = 1$ is the maximum eigenvalue and because of the symmetry of the problem it is enough to concentrate on the interval (0, 1).

Generalising slightly, one can prove the following assertion. Let $\{Q_n\}_{n=0}$ be a sequence defined by

$$Q_{n+1} = \lambda_{n+1} Q_n - \tau_n Q_{n-1}, \qquad Q_0 \neq 0 \text{ or } Q_1 \neq 0,$$

λ_n being a monotone decreasing sequence with $\lim_{n\to\infty} \lambda_n = \lambda \in (0, 1)$ and τ_n a positive sequence converging to zero. Then $Q_n \in l_2$. We will not give the details of the proof but only the steps which may be followed.

Step 1. Prove the following lemma. Let x_n be defined by

$$x_{n+1} = -\tau_n/(x_n + \lambda_n).$$

For $\varepsilon \in (0, 1)$ define $I_\varepsilon = [-(1-\varepsilon)\lambda, 0)$ and $n_0(\varepsilon)$ a positive integer such that $\tau_n < \lambda^2(1-\varepsilon)\varepsilon$ for all $n > n_0(\varepsilon)$. Then $x_m \in I_\varepsilon$ for all $m \geqslant n$ provided $x_n \in I_\varepsilon$ for $n > n_0(\varepsilon)$.

Step 2. At most a finite number of the Q_n can be zero.

Step 3. For n big enough define $x_n = Q_n/Q_{n-1} - \lambda_n$. Prove that x_n converges either to zero or to $-\lambda$. The proof proceeds by noting that if for n big enough

$$x_n \in (-\infty, -\lambda_n) \cup I_\varepsilon \cup [0, \infty)$$

then $x_m \in I_\varepsilon$ for all $m > n+1$.

Step 4. Show that $Q_n \in l_2$. This is easily done since by step 3 either $Q_n/Q_{n-1} \to \lambda \in (0, 1)$ or $Q_n/Q_{n-1} \to 0$.

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